Consider the expansion of $(7x^{12} + \frac{2}{x^8})^{20}$. **SCORE: _____ / 9 PTS**

For the questions below, you may write the coefficients in your final answers in factored form, as shown in lecture.
The coefficients in your final answers must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

NOTE: Do NOT use your calculator's ! nor $C(n, r)$ features.

$\frac{1}{2}$ POINT EACH EXCEPT AS NOTED

[a] Find the coefficient of x^{10} in the expansion.

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$$\sum_{r=0}^{20} \binom{20}{r} (7x^{12})^{20-r} (\tfrac{2}{x^8})^r = \sum_{r=0}^{20} \binom{20}{r} 7^{20-r} x^{12(20-r)} 2^r x^{-8r} = \sum_{r=0}^{20} \binom{20}{r} 7^{20-r} 2^r x^{240-20r}$$

$$x^{240-20r} = x^{10} \Rightarrow 240 - 20r = 10 \Rightarrow 24 - 2r = 1 \Rightarrow r = \tfrac{23}{2} \text{ which is not an integer}$$

No x^{10} term, so coefficient $= 0$

[b] Find the coefficient of x^{-80} in the expansion.

$$x^{240-20r} = x^{-80} \Rightarrow 240 - 20r = -80 \Rightarrow 24 - 2r = -8 \Rightarrow r = 16$$

$$\binom{20}{16} 7^{20-16} 2^{16} = \binom{20}{16} 7^4 2^{16} = \frac{20!}{16!4!} 7^4 2^{16} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!}{16! 4 \cdot 3 \cdot 2 \cdot 1} 7^4 2^{16} = 5 \cdot 19 \cdot 3 \cdot 17 \cdot 7^4 2^{16}$$

[c] Find the sixth term in the expansion.

$$\binom{20}{5} (7x^{12})^{20-5} (\tfrac{2}{x^8})^5 = \frac{20!}{5!15!} 7^{15} x^{180} 2^5 x^{-40} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 15!} 7^{15} 2^5 x^{140}$$

$$= 19 \cdot 3 \cdot 17 \cdot 16 \cdot 7^{15} 2^5 x^{140}$$

Expand and simplify $(\sqrt{t}-2t^3)^4$. **SCORE: _____ / 6 PTS**

You may write the coefficients in your final answer in factored form, as shown in lecture.

The coefficients in your final answer must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

NOTE: Do NOT use your calculator's ! nor $C(n, r)$ features.

$$(\sqrt{t})^4 \quad + \quad \overset{1}{\underbrace{4(\sqrt{t})^3(-2t^3)}} \quad + \quad \overset{1}{\underbrace{6(\sqrt{t})^2(-2t^3)^2}} \quad + \quad \overset{1}{\underbrace{4(\sqrt{t})(-2t^3)^3}} \quad + \quad (-2t^3)^4$$

$$= \underbrace{t^2}_{\frac{1}{2}} \quad - \quad \underbrace{8t^{\frac{9}{2}}}_{\frac{1}{2}} \quad + \quad \underbrace{24t^7}_{\frac{1}{2}} \quad - \quad \underbrace{32t^{\frac{19}{2}}}_{\frac{1}{2}} \quad + \quad 16t^{12} \quad 1$$

Prove that $\sum_{i=1}^{n}(2i-3)3^{i-1} = (n-2)3^n + 2$ for all positive integers n **using mathematical induction**. **SCORE: _____ / 15 PTS**

Basis case: $\sum_{i=1}^{1}(2i-3)3^{i-1} = (-1)3^0 = -1 = (-1)3^1 + 2$

GRADED BY ME

Inductive step: Assume that $\sum_{i=1}^{k}(2i-3)3^{i-1} = (k-2)3^k + 2$ for some arbitrary integer $k \geq 1$

Prove that $\sum_{i=1}^{k+1}(2i-3)3^{i-1} = (k+1-2)3^{k+1} + 2 = (k-1)3^{k+1} + 2$

$$\sum_{i=1}^{k+1}(2i-3)3^{i-1}$$

$$= \sum_{i=1}^{k}(2i-3)3^{i-1} + (2(k+1)-3)3^k$$

$$= \sum_{i=1}^{k}(2i-3)3^{i-1} + (2k-1)3^k$$

$$= (k-2)3^k + 2 + (2k-1)3^k$$

$$= (k-2)3^k + (2k-1)3^k + 2$$

$$= (k-2+2k-1)3^k + 2$$

$$= (3k-3)3^k + 2$$

$$= 3(k-1)3^k + 2$$

$$= (k-1)3^{k+1} + 2$$

So, by mathematical induction, $\sum_{i=1}^{n}(2i-3)3^{i-1} = (n-2)3^n + 2$ for all positive integers n